2016

(November)

PHYSICS

(Honours)

SIXTH PAPER

(**Mathematical Physics**)

Full Marks : 100

Pass Marks : 35

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **all** questions

1. Answer the following : $1\times4=4$

(a) What is the modulus of the complex number $f(z)=e^z=e^{x+iy}$?

(b) What is the principal argument of the complex number $C=2+2i$?

(c) What are the zeroes of the function $f(z)=z^2-1$?

(d) What do you mean by a singularity of a function?

2. Answer *either* (a) *or* (b) :

(a) If a function is defined by $f(z) = e^z$, find the sum of $f(z)$ and its complex conjugate when both of them are expressed in polar forms. Prove that the function $F(z) = \sqrt{|xy|}$ is not analytic at the origin even though Cauchy-Riemann equations are satisfied there. 2+4=6

(b) Show that the function $u(x, y) = y^3 - 3x^2y$ is a harmonic function and hence find its harmonic conjugate $v(x, y)$ such that the function $f(z) = u(x, y) + iv(x, y)$ is an analytic function. 2+4=6

3. Answer *either* (a) *or* (b) :

(a) What do you mean by a contour? Evaluate

$$\oint_C \frac{\sin z\, dz}{\left(z - \frac{\pi}{4}\right)^3}$$

where C is the circle $|z| = 3$. 2+3=5

(b) State Cauchy integral formula. Evaluate

$$\oint_C \frac{(z-1)\, dz}{(z+1)^2 (z-3)}$$

where C is the circle $|z - i| = 2$. 2+3=5

4. Answer any *three* of the following : 4×3=12

(a) Writing the principal part of the Laurent series of the function $f(z) = \dfrac{1-e^{2z}}{z^4}$, show that the singular point of this function is a pole of order 3 and also find the residue at that singularity. 3+1=4

(b) Obtain the Taylor series expansion of the function

$$f(z) = \frac{1}{(1+z)^m}$$

about the origin. 4

(c) Expand the function

$$f(z) = \frac{z}{(z-1)(z-3)}$$

in a Laurent series in the region $0 < |z-1| < 2$. 4

(d) Give a simple definition of residue at infinity of a function. Find the residue at infinity in the case of the function

$$f(z) = \frac{z^2}{z-4}$$

1+3=4

5. Evaluate any *one* of the following (using residue theorem) : 8

(i) $\int_0^\pi \frac{a\,d\theta}{a^2 + \cos^2\theta}$, $a > 0$

(ii) $\int_0^\infty \frac{dx}{x^2 + 1}$

6. Evaluate any *two* of the following (using beta and gamma functions) : 2×2=4

(i) $\int_0^\infty \frac{x^2}{(1+x)^5}\,dx$

(ii) $\int_{-1}^{+1} \sqrt{\frac{1+x}{1-x}}\,dx$

(iii) $\int_0^\infty \frac{e^{-3t}}{\sqrt{t}}\,dt$

7. Answer any *three* of the following : 3×3=9

(a) Prove that

$$P_l(-x) = (-1)^l P_l(x)$$

where l is a non-negative integer.

(b) Using the generating function for Legendre polynomials $P_l(x)$, prove that

$$\int_{-1}^{+1} \frac{P_n(x)\,dx}{\sqrt{1 - 2hx + h^2}} = \frac{2h^n}{2n+1}$$

where n is a non-negative integer.

(c) Prove that

$$P_l^m(-x) = (-1)^{l+m} P_l^m(x)$$

where l and m are positive integers.

(d) Prove that

$$\int_{-1}^{+1} P_1^1(x)\, P_1^{-1}(x)\, dx = -\frac{2}{3}$$

8. Answer *either* (a) *or* (b) :

(a) Prove the following relations : 4+3=7

(i) $e \cdot \cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} H_{2n}(x)$

(ii) $\frac{d^k}{dx^k} H_n(x) = \frac{2^k n!}{(n-k)!} H_{n-k}(x)$

where $k < n$

(b) Prove the following relations : 2+5=7

(i) $\left[\frac{d}{dx} H_{2n}(x)\right]_{x=0} = 0$

(ii) $\int_{-\infty}^{+\infty} e^{-x^2} x^2 [H_n(x)]^2\, dx = 2^n n! \left(n + \frac{1}{2}\right)\sqrt{\pi}$

9. Answer *either* (a) *or* (b) :

(a) Starting from the sum formula, derive Rodrigues' formula for generalized Laguerre polynomials, namely

$$L_n^{\alpha}(x) = \frac{x^{-\alpha} e^{x}}{n!} \frac{d^n}{dx^n}\left(e^{-x} x^{n+\alpha}\right)$$ 10

(b) Prove the following formulas :

(i) $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \cos x$

(ii) $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$

Deduce that

$$[J_{-1/2}(x)]^2 - [J_{1/2}(x)]^2 = \frac{2\cos 2x}{\pi x}$$

(4+4)+2=10

10. Answer *either (a) or (b)* :

(a) Making certain assumptions, develop the one-dimensional wave equation in the case of transverse vibrations of a light stretched string. Name the class of the partial differential equation you obtained. 9+1=10

(b) Stating the required assumptions, develop the one-dimensional equation of heat flow along a uniform solid rod. Name the class of the partial differential equation you obtained and also the SI units of thermal conductivity and diffusivity. 7+1+2=10

11. Write down the three-dimensional classical wave equation in circular cylindrical and spherical polar coordinate systems. Solve the Laplace's equation in plane polar coordinate system by the method of separation of variables. 2+8=10

12. Answer *either* (a) *or* (b) :

(a) If a function $f(x)$ is defined by

$$f(x) = \begin{cases} \pi + x, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$$

prove that its Fourier series representation is given by

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right]$$

Hence deduce that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$$ 6+1=7

(b) A periodic function is defined by

$$f(t) = f(t+T) = \begin{cases} \dfrac{at}{T}, & 0 \le t \le \dfrac{T}{2} \\ a\left(1 - \dfrac{t}{T}\right), & \dfrac{T}{2} \le t \le T \end{cases}$$

Sketch a profile of the function and its periodic extension. Also, obtain a Fourier series representation of the function. 1+6=7

13. Answer *either* (a) *or* (b) :

(a) A certain function is defined by

$$f(x) = f(x+L) = \begin{cases} -k, & -L \le x \le 0 \\ k, & 0 < x \le L \end{cases}$$

Prove that the Fourier series representation of this function is given by

$$f(x) = \frac{4k}{\pi}\left[\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \frac{1}{5}\sin\frac{5\pi x}{L} + \frac{1}{7}\sin\frac{7\pi x}{L} + \dots\right]$$

Hence deduce that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$$

5+3=8

(b) The output voltage of a half-wave rectifier of an alternating voltage source is defined by

$$V(t) = \begin{cases} V_0 \sin\omega t, & 0 \le t \le \dfrac{T}{2} \\ 0, & \dfrac{T}{2} < t \le T \end{cases}$$

where V_0 is the peak voltage and ω is the circular frequency, T is the time period of the alternating source and t is the instantaneous time. Sketch a profile of the rectified output voltage waveform and also, obtain a Fourier series representation of it.

3+5=8

★★★

Total No. of Printed Pages—9

PHY : SE H–506

2 0 1 7

(November)

PHYSICS

(Honours)

SIXTH PAPER

(Mathematical Physics)

Full Marks : 100

Pass Marks : 35

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **all** questions

1. In the study of complex variables, what do you understand by the following? 1×4=4

(a) Principal argument of a complex number

(b) Analytic part of a Laurent series

(c) An isolated singularity

(d) A pole of order m

2. *Either*

(a) What do you mean by an analytic function and an entire function? Show that the function $f(z) = z^*$ is not analytic anywhere. 1+1+4=6

Or

(b) What do you mean by a continuous function? Prove that the function $f(z) = x^2 + iy^3$ is not analytic at the origin even though Cauchy-Riemann equations are satisfied there. 2+4=6

3. *Either*

(a) For an analytic function $f(z)$, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4\,|f'(z)|^2$$ 6

Or

(b) Show that the function $v(x, y) = e^x \sin y$ is a harmonic function. Find its harmonic conjugate function $u(x, y)$ and the corresponding analytic function $f(z)$. 2+4=6

4. For an analytic function $f(z)$, prove that its nth order derivative at $z = a$ is given by

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

where C is a closed curve enclosing the point $z = a$. Using this formula, prove that

$$\oint_C \frac{e^{iz}}{z^3} dz = -\pi i$$

where C is the circle $|z| = 2$. 4+2=6

5. Answer the following :

(a) Obtain the Laurent series representing the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

valid for $1 < |z| < 2$. 3

(b) Determine the nature of singularity in respect of the function $f(z) = \dfrac{\sin z}{z}$. 2

6. Prove any *one* of the following (use Residue theorem) : 8

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2} = \frac{5\pi}{32}$

(b) $\displaystyle\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$

7. Using Beta and Gamma functions, prove the following : 2+2=4

(a) $\int_0^\infty \frac{x}{(1+x)^5} dx = \frac{1}{12}$

(b) $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

8. Answer any *three* of the following : 3×3=9

(a) Prove that

$$P_l'(1) = \frac{l(l+1)}{2}$$

where the prime denotes differentiation.

(b) Using the generating function for Legendre polynomials $P_l(x)$, prove that

$$\sum_{l=0}^{\infty} P_l(x) = \frac{1}{\sqrt{2(1-x)}}$$

(c) Prove that $P_l^l(\cos\theta) = (2l-1)!!\sin^l\theta$, where l is any non-negative integer and

$$(2l-1)!! = 1.3.5.7.\cdots(2l-1)$$

(d) Prove that

$$P_2^2(\cos\theta) = 3\sin^2\theta$$

9. *Either*

(a) Prove that for any positive integer n

$$\int_{-\infty}^{+\infty} e^{-x^2} xH_n(x)[H_{n+1}(x) - H_{n-1}(x)\,dx] =$$

$$2^n\, n!\left(n+\frac{1}{2}\right)\sqrt{\pi}$$

Also, find the value of

$$\int_{-\infty}^{+\infty} e^{-x^2} [H_2(x)]^2\, dx$$ 5+2=7

Or

(b) Prove that

$$H_n(x) = 2^n \left[\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right] x^n$$

where n is a non-negative integer. For what value(s) of n, the Hermite polynomial $H_n(0)$ will vanish? 6+1=7

10. *Either*

(a) Prove the following : 7+3=10

(i) $\int_0^{\infty} e^{-x} x^{\alpha} [L_n^{\alpha}(x)]^2\, dx = \dfrac{\Gamma(n+\alpha+1)}{n!}$

(ii) $\int_0^{\frac{\pi}{2}} \sqrt{\pi x}\, J_{\frac{1}{2}}(2x)\, dx = 1$

Or

(b) Prove : 3+7=10

(i) $\frac{d}{dx} L_n^{\alpha}(x) = -L_{n-1}^{\alpha+1}(x)$

(ii) $J_0(\sqrt{x^2 - 2xt}) = \sum_{s=0}^{\infty} J_s(x) \frac{t^s}{s!}$

11. *Either*

(a) A uniform string of length $2L$ is stretched and fastened at the ends at $x = 0$, $x = 2L$. The middle point of the string is then plucked, making an initial vertical displacement h perpendicular to the initial equilibrium position and then it is freely released from rest. Prove that the displacement of a point of the string at a distance s from the middle point at an instant of time t is given by

$$u(L \mp s, t) = \frac{8h}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \sin\left\{\frac{(2n+1)\pi}{2}\right\} \sin\left\{\frac{(2n+1)\pi(L \mp s)}{2L}\right\} \cos\left\{\frac{(2n+1)\pi v t}{2L}\right\}$$

where v is the speed of the propagation of the wave. 10

Or

(b) A homogeneous circular membrane of radius a is set to vibrations keeping the periphery at rest. Obtain the displacement $u(x, t)$ at any point distant r from the centre at an instant of time t, given that the initial displacement is $u(r, 0) = f(r)$ and the initial speed is zero. 10

12. *Either*

(a) The one-dimensional heat conduction equation is given by

$$\frac{\partial}{\partial t} u(x, t) = \kappa \frac{\partial^2}{\partial x^2} u(x, t)$$

where $u(x, t)$ is the temperature at the position x at time t and κ (kappa) is a quantity known as the diffusivity of the medium.

Determine the class of partial differential equation to which the above PDE belongs. Also, define the diffusivity of a given medium.

Solve the two-dimensional Laplace's equation in Cartesian coordinate system by the method of separation of variables. 3+2+5=10

Or

(b) Write down the three-dimensional classical wave equation in circular cylindrical and spherical polar coordinates respectively. Show that a function $\psi(\vec{r}, t) = Ae^{i(\vec{K}\cdot\vec{r} - \omega t)}$, where A is a constant, $\vec{K}$ is a constant vector called the propagation vector of the wave and ω (omega) is the angular frequency of the wave, satisfies the three-dimensional classical wave equation with the propagation speed given by

$$v = \frac{\omega}{K}$$

where $K = |\vec{K}|$. 2+2+6=10

13. *Either*

(a) A periodic function $f(x)$ is defined by

$$f(x) = f(x+2L) = x^2, \; -L \le x \le +L$$

Prove that the Fourier series representation of the function is given by

$$f(x) = x^2 = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right)$$

Hence deduce that—

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \infty = \frac{\pi^2}{12}$

(ii) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \infty = \frac{\pi^4}{90}$

6+2+2=10

Or

(b) A periodic function $g(x)$ is defined by

$$g(x) = g(x + 2L) = x, \ -L \leq x \leq +L$$

Sketch a profile of the function and show its periodic extension outside the given interval. Prove that the Fourier series representation of the function is given by

$$g(x) = x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)$$

Hence deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \infty = \frac{\pi}{4}$$ 3+5+2=10

14. Prove the Parseval's formula that can be expressed as

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$ 5

★★★

PHY: SE –H 506

2021

(November)

PHYSICS

(Honours)

SIXTH PAPER

(Mathematical Physics)

Section – A for Regular Candidates

Full Mark: 70

Pass Mark: 25

Section – A and Section – B for Back/Casual Candidates

Full Mark: 100

Pass Mark: 35

Time: 3 hours

The figures in the margin indicate full marks for the questions.

Section – A

1. a) What do you mean by a harmonic function? 1

   b) What is the Principal argument of the complex number, Z=3+3i? 1

   c) Define a simple pole. 1

   d) For what values of n the Hermite polynomial Hn(o) will vanish? 1

2. a) For an analytic function f(z), prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f(z)'|^2$ 6

   Or

   b) If a function is define by f(z)=e^z, find the sum of f(z) and its complex conjugate when both of them are expressed in polar forms.

   Prove that the function $F(Z)=\sqrt{|xy|}$ is not analytic at the origin even though Cauchy-Riemann equations are satisfied there. 2+4=6

3. State and prove Cauchy's Theorem of an analytic function of a complex variable. 1+5=6

4. A) For an analytic function f(z), Prove that its nth order derivation at Z=a is given by $F_a^n=\frac{n!}{2\lambda 2}\oint_c\frac{f(z)dz}{(z-a)^{n+1}}$ where C is a closed curve enclosing the point Z=a. 4

   b) Using this formula prove that $\oint_c\frac{e^{2z}}{Z^3}dz=-\lambda i$ where C is the circle $|Z|=2$. 3

5. Using residue theorem prove that $\int^{2\lambda}\frac{d\theta}{(a+b\cos\theta)^2}=\frac{2\lambda a}{(a^2-b^2)^{\frac{3}{2}}}$ where a>b>0. 8

6. a) For a non negative integer n, prove that, $\int_{-1}^{+1}\frac{Pn^{(x)}dx}{\left(\sqrt{\left(1-2xh+h^2\right)}\right)} = \frac{2h^n}{2n+1}$ 4

b) Prove that $P_l^m(-x) = (-1)^{l+m} P_l^m(x)$ where 'l' and 'm' are positive integers. 4

7. a) Show that $\int_{-1}^{+1} xP_l(x)P_{l-1}(x)dx = \frac{2l}{4l^2-1}$ 4

b) Prove that $\int_{0}^{\frac{\lambda}{2}} \sqrt{(\lambda x)}\, J_{\frac{1}{2}}(2x)dx = 1$ 3

8. a) in the case of Bessel function's of the first kind, derive the recurrence relation $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$ 4

Or

b) Prove that $\frac{J_n(x)}{X^n} = \frac{1}{2^{n\Gamma(n+1)}}$ 3

i) $\lim x \to 0$

ii) $L_n(0) = 1$ 1

9. a) Stating the required assumptions develop the one dimensional equation of heat flow along a uniform solid bar. Name the class of the partial differential equations you obtained and also the SI units of thermal conductivity and diffusivity. 7+1+2=10

b) Making certain assumptions develop the one dimensional wave equation in the case of transverse vibrations of a light stretched string. Name the class of the partial differential equation you obtained. 9+1=10

10. A certain functions is defined by

$f(x) = f(x+\lambda) = \begin{cases} -k, -\lambda \le x \le 0 \\ +k, 0 < x \le \lambda \end{cases}$ and deduce that $\frac{\lambda}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ 4+1=5

11. Prove that a complex form of Fourier Series is equivalent to a Fourien series. 5

Section-B

12. Using gamma functions,

a) Prove that $\Gamma(n) = (n-1)!$, when 'n' is a positive integer.

b) Prove that $\Gamma(-\frac{1}{2}) = -2\sqrt{\lambda}$

Or

If a function f(x) is defined by $f(x) = \begin{cases} \lambda + x, -\lambda \le x \le 0 \\ \lambda - x, 0 \le x \le \end{cases}$

Hence, deduce $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots = \frac{\lambda^2}{8}$. 5

13. For a Bessel function, show that $\sqrt{\frac{\lambda x}{2}}\, J_{\frac{3}{2}}(x) = \frac{1}{x}\sin x - \cos x$ 5

14. Answer either (a) and (b) or (c).

Find the residues of the following functions f(z) at the given poles.

a) $f(z) = \frac{SinZ}{1 - Z^4}$ at Z=I 3+2=5

b) $f(z) = \frac{SinZ}{(Z - \lambda)^3}$ at $Z = \lambda$

Or

c) Prove that $\Gamma(n)\Gamma(1-n) = \frac{\lambda}{\sin(n\lambda)}$, *for* $0 < n < 1$ when 'n' is an integer. 5

15. Show that

a) $\cos x = J_0 - 2J_1 + 2J_4$

b) $Sinx = 2J_1 - 2J_3 + 2J_5$ 5

16. If alight stretched string of length 'l' is plucked at its midpoint by the displacement 'h'. Show that the deflection of the string at a distance 'x' from one end at time 't' is given by

$$U(x,t) = \frac{8h}{\lambda^2}\left(\frac{\sin \lambda x}{l}, \frac{\cos \lambda \vartheta t}{l} - \frac{1}{9}\sin\frac{3\lambda x}{l}.\cos\frac{3\lambda \vartheta t}{l} + \ldots\ldots \right)$$

Where ϑ is the constant speed of propagation of wave along the string. Give the reason why vibrations of even harmonics are absent. 9+1=10

****

5th Semester Exam., 2022 (November)

PHYSICS

(Honours)

SIXTH PAPER

(Mathematical Physics)

Full Marks : 100

Pass Marks : 35

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×4=4

(a) What do you mean by delta neighbourhood?

(b) Convert

$$3e^{-i\tan^{-1}\left(\frac{3}{2}\right)}$$

to rectangular coordinate.

(c) What is a Jordan curve?

(d) What is the generating function of Laguerre polynomial?

2. Verify Cauchy's theorem for the function

$$f(z) = z^3 - iz^2$$

on the circle $C: |z| = 1$. 3

3. *(a)* Find the conjugate function of

$$u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$$ 5

Or

(b) Find the analytic function whose real part is $\cos x \cosh y$.

4. *(a)* If

$$A(x, y) = xy - ix^2y^3$$

find grad A, div A and curl A. 2+2+2=6

Or

(b) Prove that div grad $A = 0$, if A is imaginary or more generally if $\text{Re}\{A\}$ is harmonic. 3+3=6

5. *(a)* Evaluate : 8

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3} dx$$

Or

(b) Evaluate :

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)(x^2+4)}$$

Or

(c) Show that the function

$$f(z) = \begin{cases} \dfrac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2} &, \quad z \neq 0 \\ 0 &, \quad z = 0 \end{cases}$$

satisfies Cauchy-Riemann condition at $z = 0$. Determine whether the function is analytic at $z = 0$. 4+4=8

6. Evaluate : 5

$$\int_0^1 (1 - x^n)^{1/n} dx$$

7. (a) Prove the following : 2½+2½=5

(i) $J_{\frac{1}{2}}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$

(ii) $J_{-\frac{1}{2}}(x) = \sqrt{\dfrac{2}{\pi x}} \cos x$

Or

(b) Prove that

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

with $n \geq 1$. 5

Or

(c) Prove that

$$x\frac{d}{dx}L_n^k(x) = nL_n^k(x) - (n+k)L_{n-1}^k(x)$$ 5

8. (a) Expand the function

$$f(x) = x^4 - 2x^3 + 3x^2 + 5x - 9$$

in a series of Legendre polynomials. 8

Or

(b) Prove that

$$P_l^{m+1}(x) - \frac{2mx}{\sqrt{1-x^2}}P_l^m(x) +$$
$$\{l(l+1) - m(m-1)\}P_l^{m-1}(x) = 0$$

9. (a) Write down the equation satisfied by free transverse vibration of a string with ends fixed and solve it. 10

Or

(b) There is a rectangular membrane with length l and breadth b. The boundary of the membrane is fixed. If the initial displacement is

$$\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{3\pi y}{b}\right)$$

and initial velocity is zero, find the displacement of the membrane at an arbitrary time t.

10. *(a)* Solve the Laplace's equation in plane polar coordinate system by the method of separation of variables. 5

Or

(b) Develop the one-dimensional equation of heat flow along a uniform solid rod.

11. *(a)* Find the Fourier series representing $f(x) = x,\ 0 < x < 2\pi$ and sketch its graph from $x = -4\pi$ to $x = 4\pi$. 5+1=6

Or

(b) Find the Fourier series representing $f(x) = x^2,\ -\pi < x < \pi$ and sketch its graph from $x = -2\pi$ to $x = 2\pi$. 5+1=6

12. Find the Fourier cosine series for the function

$$f(x) = \begin{cases} 1 \text{ for } 0 < x < \pi/2 \\ 0 \text{ for } \pi/2 < x < \pi \end{cases}$$ 5

13. Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. 3

14. *(a)* Find the Taylor series expansion of

$$f(z) = \frac{1}{(1+z)^2}$$

about the origin. 5

Or

(b) Find the Laurent series expansion of

$$f(z) = \frac{e^{2z}}{(z-1)^3}$$

about $z = 1$.

15. *(a)* Find the residue of

$$f(z) = \frac{e^{2z}}{1+e^z}$$

at its pole. 5

Or

(*b*) Find the residue of

$$f(z) = \frac{1}{z^3 - z^5}$$

at infinity.

16. (*a*) Find the solution of

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

about $x = 0$ by using Frobenius method. 8

Or

(*b*) Prove that

$$1 + \frac{1}{2}P_1(\cos\theta) + \frac{1}{3}P_2(\cos\theta) + \cdots = \ln\left(\frac{1 + \sin\theta/2}{\sin\theta/2}\right)$$

17. A rod of length l with insulated sides is initially at a temperature $T(x, 0) = A$. Its ends are kept at constant temperature 0 °C. Find the temperature $T(x, t)$ at a point distant x after time t. 5

18. Find the Fourier sine series for the function $f(x) = e^x$ for $0 < x < \pi$. 4

★ ★ ★