NAMBOL L. SANOI COLLEGE, NAMBOL

DEPARTMENT OF MATHEMATICS

QUESTION BANK FOR MATHEMATICS

PREVIOUS 4 YEARS (2016-2019)

SEMESTER V

OBJECTIVE TYPE

(1 MARK QUESTIONS)

2016

MATHEMATICS

(Honours)

SIXTH PAPER

(Real Analysis)

1. Choose and rewrite the correct answer for each of the following questions : 1×5=5

(a) For a sequence $\{u_n\}$, where

$$u_n = \begin{cases} 0 & \text{if } n = 4k \\ 3 & \text{if } n = 4k+1 \\ -7 & \text{if } n = 4k+2 \\ 5 & \text{if } n = 4k+3, \ k = 0, 1, 2, 3, \cdots \end{cases}$$

the values of $\overline{\lim} u_n$ and $\underline{\lim} u_n$ are respectively

(i) 0 and 3

(ii) 3 and 0

(iii) −7 and 5

(iv) 5 and −7

(b) If f is a bounded function defined on $[a, b]$ and P is a partition of $[a, b]$ and P^* is the refinement of P, then the false result is

(i) $L(P, f) \le L(P^*, f)$

(ii) $L(P, f) \le U(P, f)$

(iii) $U(P, f) \le U(P^*, f)$

(iv) $U(P, f) \ge U(P^*, f)$

(c) Convergence in $\int_0^1 \frac{1}{x^p} dx$ and $\int_1^\infty \frac{1}{x^q} dx$ are for what values of p and q?

(i) $p < 1$ and $q < 1$

(ii) $p < 1$ and $q > 1$

(iii) $p > 1$ and $q > 1$

(iv) $p > 1$ and $q < 1$

(d) If $f(x, y)$ is differentiable at a point (2, 3), then the false conclusion is

(i) f is continuous at (2, 3)

(ii) f is discontinuous at (2, 3)

(iii) f_x and f_y exist at (2, 3)

(iv) simultaneous limit of f exists at (2, 3)

(e) Stokes' theorem gives means of converting

(i) a line integral to a double integral

(ii) a line integral to a surface integral

(iii) a line integral to a triple integral

(iv) a surface integral to a triple integral

2018

(a) For the sequence $\{u_n\}$ where

$$u_n = \begin{cases} 3, & \text{if } n = 3k \\ -5, & \text{if } n = 3k-1 \\ 4, & \text{if } n = 3k-2, \quad k \geq 1 \end{cases}$$

The values of the $\underline{\lim}\, u_n$ and $\overline{\lim}\, u_n$ are respectively

(i) 3 and 4 (ii) 4 and −5

(iii) −5 and 4 (iv) 4 and 3

(b) If $f(x) = 3x - 2$ and $p = \{0, 3, 4, 6\}$ is a partition of $[0, 6]$, then the value of oscillatory sum $w(p, f)$ is

(i) 26

(ii) 42

(iii) 24

(iv) 62

(c) The improper integrals

$$\int_1^\infty \frac{1}{x^m}\,dx \text{ and } \int_0^1 \frac{1}{x^n}\,dx$$

are both convergence if

(i) $m > 0$ and $n > 0$

(ii) $m < 1$ and $n > 1$

(iii) $m < 1$ and $n < 1$

(iv) $m > 1$ and $n < 1$

(d) If f_{xy} and f_{yx} are both continuous at (1, 2), then

(i) $f_{xy}(1, 2) = f_{xx}(1, 2)$

(ii) $f_{xy}(1, 2) = f_{yy}(1, 2)$

(iii) $f_{xx}(1, 2) = f_{yy}(1, 2)$

(iv) $f_{xy}(1, 2) = f_{yx}(1, 2)$

(e) The value of the triple integral

$$\iiint_V xyz\,dx\,dy\,dz$$

where V is the parallelopiped $[0, 4 : 0, 3; 0, 2]$ is

(i) 12

(ii) 36

(iii) 72

(iv) 576

(a) The number of limit points of the sequence where

$$u_n = \begin{cases} 20 & , \text{ for } n = 1 \\ \text{least prime factor of } n, & \text{for } n > 1 \end{cases}$$

is

(i) 0 *(ii)* 20

(iii) two only *(iv)* infinite

(b) If $[x]$ denotes the greatest integer not exceeding x, then $\int_1^3 [x]\, dx$ equals

(i) 1

(ii) 2

(iii) 3

(iv) 4

(c) The number of points of infinite discontinuity of the improper integral

$$\int_{-4}^{6} \frac{(x-4)\, dx}{(x^2-9)(x^2-4)(x^2-5x+6)(x^2+5x-6)}$$

is

(i) 3

(ii) 4

(iii) 5

(iv) 6

(d) If $\phi(x, y)$ is differentiable at a point (a, b), then the false conclusion is

(i) $\phi(x, y)$ is continuous at (a, b)

(ii) $\phi(x, y)$ is discontinuous at (a, b)

(iii) ϕ_x and ϕ_y exist at (a, b)

(iv) simultaneous limit of ϕ exists at (a, b)

(e) The length of the curve $x = a\cos\theta$, $y = a\sin\theta$, $z = a\theta$ from $\theta = 0$ to $\theta = 2\pi$ is

(i) $a\pi$

(ii) $\sqrt{2}a\pi$

(iii) $2a\pi$

(iv) $2\sqrt{2}a\pi$

VERY SHORT ANSWER TYPE

(1 MARK QUESTIONS)

2016

2. Write very short answer for each of the following questions : $1 \times 8 = 8$

(a) State nested interval theorem.

(b) When is a function f said to be discontinuity of second kind at $x = a$?

(c) If $f(x) = x - 6$ and $p = \{0, 3, 4, 6\}$ is a partition $[0, 6]$, then find the value of $w(p, f)$.

(d) Write the value of $\Gamma\left(\frac{3}{2}\right)$.

(e) State Dirichlet's test for convergence of improper integral.

(f) State Young's theorem on the reversal of order of partial derivation.

(g) If $z = e^{xy^2}$, $x = t\cos t$, $y = t\sin t$, find $\frac{dz}{dt}$.

(h) State Stokes' theorem.

2017

(a) Give an example of a sequence which has an infinite number of limit points.

(b) Give an example of a continuous function which is not bounded in a finite open interval.

(c) Give an example of a non-integrable function f such that $|f|$ is integrable.

(d) State Frullani's improper integral theorem.

(e) State Abel's test (for convergence of improper integral).

(f) When is a function said to be differentiable at a point (x, y)?

(g) Find $\frac{\partial^2 z}{\partial x \partial y}$, when $z = 2x^3y^4$.

(h) If $D = \{(x, y) : x^2 + y^2 \leq 3\}$, then express

$$\iint_D f(x, y)\,dxdy$$

as a repeated integral.

2018

(a) Define an open set.

(b) When is a function said to be discontinuity of the second kind at $x = a$?

(c) Is the sum of two non-integrable functions necessarily non-integrable? Give an example in support of your answer.

(d) Write all the points of infinite discontinuity of the integrand of the integral

$$\int_0^7 \frac{1}{(x^2-1)(x^2-4)(x^2-5x-6)(x^2-5x+6)}\,dx$$

(e) State Dirichlet's test for convergence of improper integral.

(f) When is a function said to be differentiable at the point (x, y)?

(g) Examine the equality of f_{xy} and f_{yx}, where $f(x, y) = e^{2xy}$.

(h) If $D = \{(x, y) : x^2 + y^2 \le 12\}$, then express the double integral

$$\iint_D f(x, y)\,dx\,dy$$

as a repeated integral.

2019

(a) Define limit point of a given set S of real numbers.

(b) Verify by giving an example that a function continuous in an open interval need not be bounded therein.

(c) Is the product of two non-integrable functions necessarily non-integrable? Give an example to support your answer.

(d) For what values of n the improper integral $\int_1^\infty \frac{1}{x^n}\,dx$ convergent?

(e) State Abel's test (for convergence of improper integral).

(f) Find $\frac{\partial^3 z}{\partial x\,\partial y^2}$, where $z = x^4 y^3$.

(g) If $z = x^3 - xy + y^3$, $x = r\cos\theta$ and $y = r\sin\theta$, then find $\frac{\partial z}{\partial r}$.

(h) If $D = \{(x, y) : x^2 + y^2 \le 8\}$, then express

$$\iint_D f_{(x,\ y)}\,dx\,dy$$

as repeated integral.

SHORT ANSWER TYPE

(3 MARKS QUESTIONS)

2016

3. Write short answer for each of the following questions : 3×13=39

(a) Prove that the union of an arbitrary family of open sets is open. Is arbitrary intersection of open sets necessarily open? Give an example.

(b) Show that every bounded sequence has a limit point.

(c) Show that the sequence $\{\delta_n\}$, where

$$\delta_n = \left(1+\frac{1}{n}\right)^n$$

is convergent and that $\lim\left(1+\frac{1}{n}\right)^n$ lies between 2 and 3.

(d) Let f be a function on R defined by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational} \end{cases}$$

Show that f is discontinuous at every point of R.

(e) If a and b be any two positive real numbers, then show that there exists a positive integer n such that $na > b$.

(f) Prove that every monotonic function is R-integrable.

(g) If a function f is bounded and integrable on each of the interval $[a, c]$, $[c, b]$, $[a, b]$, where c is a point of $[a, b]$, then prove that

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$$

(h) Test the convergence of the improper integral

$$\int_3^5 \frac{x^2 dx}{\sqrt{(x-3)(5-x)}}$$

(i) By using Frullani's improper integral theorem, prove that

$$\int_0^\infty \frac{\cos ax - \cos bx}{x} dx = \log\left(\frac{b}{a}\right)$$

where $a, b > 0$ and deduce that

$$\int_0^\infty \frac{\sin ax \sin bx}{x} dx = \frac{1}{2}\log\left(\frac{a+b}{a-b}\right)$$

(j) Show that the function f, where

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x = y = 0 \end{cases}$$

is not differentiable at the origin.

(k) If

$$f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$.

(l) Evaluate by changing the order of integration

$$\int_0^{2a} dx \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f dy$$

(m) Find the surface area of that part of the cone $z^2 = x^2 + y^2$ which lies inside the cylinder $x^2 + y^2 = 2x$.

2017

(a) Define an open set. Show that every open interval is an open set.

(b) The derived set of every set is a closed set.

(c) Prove that bounded and monotonically decreasing sequence converges to its infimum.

(d) Define subsequence of a sequence. Show that every subsequence of a convergent sequence converges to the same limit.

(e) Let f be defined on Q (the set of all rational numbers) in the following manner :

$$f(x) = \begin{cases} 5 & \text{if } x < 2\pi \\ -3 & \text{if } x > 2\pi, x \in Q \end{cases}$$

Show that f is continuous on Q but not uniformly on Q.

(f) If $f(x) = 2\sin x$ and $P = \{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{5\pi}{6}\}$ be a partition of $[0, \frac{5\pi}{6}]$, find—

(i) the oscillation of the function on $[0, \frac{5\pi}{6}]$;

(ii) $U(P, f)$;

(iii) $L(P, f)$.

[The symbols have their usual meanings.]

(g) If f is bounded and integrable on $[a, b]$ and $\phi(x) = \int_a^x f(t)\,dt \;\; \forall x \in [a, b]$, prove that ϕ is continuous on $[a, b]$.

(h) Examine the convergence of the improper integral

$$\int_0^3 \frac{dx}{x^{2/3}(3-x)^{1/4}}$$

(i) Show that $\int_0^{\pi/2} \sin x \log \sin x \, dx$ is convergent with the value $\log(2/e)$.

(j) If $f(x, y) = \frac{3xy(x^2 - y^2)}{x^2 + y^2}$, $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$, then find $f_{yx}(0, 0)$.

(k) If $u = F(x, y, z)$ and $z = f(x, y)$, then find the formula for $\frac{\partial^2 u}{\partial x^2}$ in terms of the derivatives of F and the derivatives of z (assuming F and f are differentiable).

(l) Using the line integral, find the area of the ellipse $x = a\cos\theta$, $y = b\sin\theta$.

(m) Evaluate the surface integral

$$\iint_S (xdydz + ydzdx + zdxdy)$$

with the help of Gauss' theorem over the surface of the parallelopiped [0,3; 0,4; 0,2].

2018

(a) If a and b be any two positive real numbers, then show that there exists a positive integer n such that $na > b$.

(b) Show that the complement of an open set is a closed set.

(c) Every convergent sequence is a Cauchy sequence but not the converse. Prove this.

(d) Show that the sequence $\{u_n\}$ where

$$u_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$

is convergent.

(e) Prove that a function which is uniformly continuous on $[a, b]$ is continuous on $[a, b]$.

(f) Prove that every continuous function on $[a, b]$ is R-integrable on $[a, b]$.

(g) If

$$\int_a^b f(x)\,dx$$

exists, then show that

$$\left|\int_a^b f(x)\,dx\right| \le \left|\int_a^b |f(x)|\,dx\right|$$

(h) Test the convergence of

$$\int_0^1 \log x^5\,dx$$

(i) Test the convergence of

$$\int_{\pi/2}^{\pi} \frac{\sqrt{x}}{\sin x}\,dx$$

(j) Show that the function f, where

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

is not differentiable at the origin.

(k) If $f(x, y, z) = (x^2+y^2+z^2)^{-\frac{1}{2}}$, show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

(l) By Changing the order of integration, prove that

$$\int_0^t dx \int_0^x f(y)\,dy = \int_0^t (t-y)\,f(y)\,dy$$

(m) Find the value of

$$\int (x^2 y\,dx - y^2 x\,dy)$$

taken along the circle $x^2+y^2=4$ in the counter-clockwise sense.

2019

(a) Define an open set and show that the arbitrary union of open sets is open.

(b) Prove that derived set of every set is a closed set.

(c) Every convergent sequence is a Cauchy sequence but not the converse. Prove it.

(d) Prove that a bounded and monotonically decreasing sequence converges to its greatest lower bound.

(e) If a function f is continuous in the closed interval $[a, b]$ and $f(a) \neq f(b)$, then prove that f assumes every value between $f(a)$ and $f(b)$.

(f) Show that if f is monotonically increasing in $[a, b]$, then it is integrable in $[a, b]$.

(g) State and prove the fundamental theorem of integral calculus.

(h) Examine the convergence of the improper integral

$$\int_0^2 \frac{dx}{x^{\frac{1}{4}}(2-x)^{\frac{2}{3}}}$$

(i) Show that $\int_0^{\pi/2} \sin x \log \sin x \, dx$ is convergent with the value $\log(2/e)$.

(j) If

$$f(x, y) = \frac{2xy(x^2 - y^2)}{x^2 + y^2}, \ (x, y) \neq (0, 0)$$

and $f(0, 0) = 0$, then find $f_{xy}(0, 0)$.

(k) If (i) x, y be differentiable functions of a single variable t and (ii) z is differentiable function of x and y, then show that z possesses continuous derivative with respect to t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

(l) By using Green's theorem on plane, convert the line integral

$$\int_C (x^2 y \, dx + xy^2 \, dy)$$

taken along the unit circle $x^2 + y^2 = 1$, into a double integral and hence evaluate it.

(m) Prove by using Gauss divergent theorem that

$$\iint_S (x \, dy \, dz + y \, dz \, dx + z \, dx \, dy) = 81$$

where S is the outer surface of the cube

$$[0, 3; \ 0, 3; \ 0, 3]$$

(6 MARKS QUESTIONS)

2016

4. (a) State and prove Cauchy's general principle of convergence (for sequence). 1+5=6

Or

State and prove Bolzano-Weierstrass theorem (for sets). 1+5=6

(b) Prove that every uniformly continuous function is continuous on that interval. By giving a suitable example, show that the converse of the above statement is not true. 6

Or

Prove that if a function f is continuous in $[a, b]$, then it is bounded.

5. (a) State a necessary and sufficient condition for Riemann integrability of a bounded function f on an interval $[a, b]$ and prove the same. 1+5=6

Or

Prove that the oscillation of a bounded function f on an interval $[a, b]$ is the supremum of the set

$$\{|f(x_1) - f(x_2)| : x_1, x_2 \in [a, b]\}$$ 6

(b) If f is bounded and integrable on $[a, b]$, then prove that $|f|$ is also bounded and integrable on $[a, b]$ and

$$\left|\int_a^b f dx\right| \le \int_a^b |f| \, dx$$ 6

Or

If f be a bounded function on $[a, b]$ and f has a finite number of limit points of its points of discontinuity, then prove that f is integrable on $[a, b]$.

6. (a) Show that the integral

$$\int_0^{\pi/2} \log \sin x \, dx$$

is convergent and evaluate it. 6

Or

Test the convergence of the improper integral $\int_0^{\infty} e^{-x} x^{x-1} dx$.

(b) Show that for the function

$$f(x, y) = xy\frac{(x^2 - y^2)}{x^2 + y^2}, \ (x, y) \ne (0, 0)$$

$f(0, 0) = 0$ does not satisfy the condition of Schwarz's theorem and

$$f_{xy}(0, 0) \ne f_{yx}(0, 0)$$

Or

If z is a function of u and v and $u = x^2 - y^2$, $v = 2xy$, prove that

$$4(u^2 + v^2)\frac{\partial^2 z}{\partial u \partial v} + 2u\frac{\partial z}{\partial v} + 2v\frac{\partial z}{\partial u} = xy\left(\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2}\right) + \frac{1}{2}(x^2 - y^2)\frac{\partial^2 z}{\partial x \partial y}$$

7. *(a)* Design a neat diagram of the region of integration and change the order of integration in the double integral

$$\int_0^1 dx \int_x^{1/x} \frac{y\,dy}{(1+xy)^2(1+y^2)}$$ 6

Or

Design a neat diagram of the region of integration and change the order of integration in the double integral

$$\int_0^{2a} \int_{x^2/4a}^{3a-x} f(x, y)\,dx\,dy$$

and verify the result when $f(x, y) = 3$.

(b) Use Gauss theorem to evaluate the integral

$$\iint_S (y^2z^2\,dydz + z^2x^2\,dzdx + x^2y^2\,dxdy)$$

where S is the surface of the sphere $x^2+y^2+z^2=1$ above the xy-plane. 6

Or

Show that

$$\iiint_E (ax+by+cz)^2\,dxdydz = \frac{4}{15}\pi(a^2+b^2+c^2)$$

where E is the sphere $x^2+y^2+z^2 \le 1$.

2017

4. *(a)* If S is a closed and bounded subset of $\mathbb{R}$ and $\mathscr{F}$ is an arbitrary open cover of S, then show that $\mathscr{F}$ has a finite subcover of S. 6

Or

State and prove nested interval theorem. 1+5=6

(b) Show that the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{when } x \text{ is rational} \\ -x, & \text{when } x \text{ is irrational} \end{cases}$$

is continuous at $x = 0$ only. 6

Or

Show that the function $f(x) = 3x^2$ is uniformly continuous on $[-5, 4]$. Also show that f is continuous on $\mathbb{R}$ (the set of real numbers) but not uniformly continuous on $\mathbb{R}$. 2+2+2=6

5. (a) State and prove one of the Darboux's theorems. 1+5=6

Or

If f is non-negative continuous function on $[a, b]$ and $\int_a^b f(x)\,dx = 0$, then prove that $f(x) = 0$ for all $x \in [a, b]$. 6

(b) If f and g are two functions of both bounded and integrable on $[a, b]$, then prove that $f + g$ is also bounded and integrable on $[a, b]$ and

$$\int_a^b \{f(x) + g(x)\}\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$ 6

Or

State and prove Bonnet's form of second mean-value theorem of integral calculus.

6. (a) Show that $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ exists if and only if both m and n are positive. 6

Or

State Dirichlet's test for improper integral. By using the same, show that the integral $\int_3^\infty \frac{4\sin x}{x^p}\,dx$ converges absolutely for $p > 1$ but only conditionally for $0 < p \le 1$.

(b) State and prove Young's theorem for the equality of $f_{xy}(a, b)$ and $f_{yx}(a, b)$. 6

Or

If $x = u + v$, $y = uv$ and z is a function of x and y, then prove that

$$\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = (x^2 - 4y)\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial z}{\partial y}$$

7. (a) State and prove Green's theorem in plane. 6

Or

Design a neat diagram of the region of integration and change the order of integration in the double integral

$$\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{a^2 - x^2}} f(x, y)\,dx\,dy$$

Verify the result when $f(x, y) = 1$.

(b) Show that

$$\iint_S (xy\,dxdy + yz\,dydz + zx\,dzdx) = \frac{3}{8}$$

where S is the surface of the sphere $x^2+y^2+z^2=1$ in the first octant. 6

Or

Show that

$$\iiint (x+y+z)\,xyz\,dxdydz = \frac{1}{840}$$

the integration being taken throughout the region bounded by the planes $x=0$, $y=0$, $z=0$, $x+y+z=1$.

2018

4. *(a)* **State and prove Bolzano-Weierstrass theorem for set.** **6**

Or

Show that a necessary and sufficient condition for the sequence $\{u_n\}$ to be convergent is that to each $\varepsilon > 0$ there exists an integer m such that

$$|u_{n+p} - u_n| < \varepsilon \,\forall n \geq m,\ p \geq 0\,(p \in \mathbb{Z})$$

(b) **If a function f is continuous on a closed interval $[a, b]$, then prove that it attains its bounds at least once in $[a, b]$.** 6

Or

If f is a continuous function in $[a, b]$ and $f(a)$, $f(b)$ are of opposite signs, then there exists a point $c \in]a, b[$ such that $f(c) = 0$. Prove this.

5. *(a)* Let $|f(x)| \leq k$ for all x in $[a, b]$ and P be a partition of $[a, b]$ with norm $\leq \delta$. If P^* is a refinement of P containing at most p more points than P, prove that

$$U(P^*, f) \subseteq U(P, f) \leq U(P^*, f) + 2pk\delta \qquad 5$$

Or

Prove that the oscillation of a bounded function f on an interval $[a, b]$ is the supremum of the set

$$\{|f(x_2) - f(x_1)| : x_1, x_2 \in [a, b]\}$$

(b) When is a bounded function f said to be integrable on $[a, b]$? State a necessary and sufficient condition for integrability of a bounded function f on an interval $[a, b]$ and prove the same. 1+1+5=7

Or

If a function f is bounded in $[a, b]$ and the set of points of discontinuity has a finite number of limit points, show that f is R-integrable on $[a, b]$. 7

6\. (a) Test the convergence of the improper integral

$$\int_0^\infty x^{n-1}e^{-x}dx$$ 6

Or

Show that the integral $\int_0^\infty \frac{\sin x}{x}dx$ converges but not absolutely.

(b) **State and prove Schwarz's theorem on the reversal of the order of partial derivation.** 6

Or

Prove that by the transformation $u = x - at$, $v = x + at$ the partial differential equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{dx^2}$$

reduces to $\frac{\partial^2 z}{\partial u \partial v} = 0$.

7\. (a) Design a neat diagram of the region of integration and change the order of integration in the double integral

$$\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x, y)\,dx\,dy$$ 4

Or

Compute the surface area of the sphere

$$x^2 + y^2 + z^2 = a^2$$

(b) State and prove Stokes' theorem *or* Gauss divergence theorem. 8

★★★

2019

4\. (a) State and prove nested interval theorem. 1+5=6

Or

State and prove Heine-Borel theorem. 1+5=6

(b) Show that the function f defined on R by

$$f(x) = \begin{cases} -x, & \text{when } x \text{ is rational} \\ x, & \text{when } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$. 6

Or

If a function f is continuous in $[a, b]$, then prove that it is uniformly continuous in $[a, b]$.

5. (a) State and prove Darboux's upper R-integral theorem. 1+5=6

Or

If f is non-negative continuous function on $[a, b]$ and $\int_a^b f(x)\,dx = 0$, then prove that $f(x) = 0$ for all $x \in [a, b]$. 6

(b) The set of points of discontinuity of a bounded function defined on $[a, b]$ is finite. Then prove that f is R-integrable on $[a, b]$. 6

Or

State and prove Bonnet's form of second mean-value theorem of integral calculus. 1+5=6

6. (a) Show that

$$\int_0^1 \frac{dx}{1+x^4 \sin x}$$

is convergent. 6

Or

State and prove the Frullani's integral theorem. 1+5=6

(b) State and prove Young's theorem for the equality of $f_{xy}(a, b)$ and $f_{yx}(a, b)$. 1+5=6

Or

Given that z is a function of x and y and that $x = u^2 v$ and $y = uv^2$, then prove that

$$2x^2 \frac{\partial^2 z}{\partial x^2} + 5xy \frac{\partial^2 z}{\partial x \partial y} + 2y^2 \frac{\partial^2 z}{\partial y^2} = uv \frac{\partial^2 z}{\partial u \partial v} - \frac{2}{3}\left(u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v}\right)$$ 6

7. (a) Design a neat diagram of the region of integration and change the order of integration in the double integral

$$\int_0^{2a} \int_{x^2/4a}^{3a-x} f(x, y)\,dx\,dy$$

and verify the result when $f(x, y) = 4$. 6

Or

Find the surface area of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ inside the cylinder $x^2 + y^2 = ax$.

(b) Evaluate $\iint_S (yz\,dy\,dz + zx\,dz\,dx + xy\,dx\,dy)$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant. 6

Or

Find the volume of the solid bounded by the surface $z = 1 - 4x^2 - y^2$ and the plane $z = 0$.

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